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Circulant covers of trivalent circulants

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Abstract

Given two graphs G_1 and G_2, one may ask whether or not G_2 is a *cover* of G_1. Feng and Kwak [Typical circulant double coverings of a circulant graph, Discrete Math. 277 (2004) 73–85] provide a description of *typical* covers of a circulant graph by another circulant graph, and use other techniques to show that there are no double covers of trivalent circulant graphs. The purpose of this paper is to provide a description of all circulant covers over trivalent circulant graphs.

Keywords: Circulant covers; Circulant graphs

1. Introduction

The terminology and notation in this paper follows that found in Feng and Kwak [1]. Throughout, all graphs under consideration will be simple, finite and undirected. For convenience, we will consider each undirected edge $\{u, v\}$ to consist of two directed arcs, namely (u, v) and (v, u). If G is a graph, then we will denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. The *open neighborhood* of a vertex $v \in V(G)$, denoted $N(v)$ is the set of all vertices adjacent to v in G, and $N[v] = N(v) \cup \{v\}$ is the *closed neighborhood* of v. We will use standard notation, K_n is the complete graph on n vertices, $K_{n,m}$ is the complete bipartite graph with sides of size n and m, and C_k is the cycle on k vertices. Further, the primary graphs under consideration will be *Cayley* graphs over finite Abelian groups. We will say that G *contains an* H if there is some subgraph of G isomorphic to H (not necessarily induced). Further, G will be called *triangle free* if G does not contain a K_3. For reasons which may become clear later, we will refer to subgraphs isomorphic to C_4 as *rectangles*.

Let $(A, +)$ be a finite Abelian group with identity 0_A (or 0 if A is clear from the context), and $X \subset A$ such that $X = -X$ and $0 \notin X$. Then $G := \mathbf{Cay}(A, X)$ will denote the Cayley graph of A with respect to the *generating set* X. Here, $V(G) = A$, and $E(G) = \{\{a, a + x\} | a \in A, x \in X\}$. From the definitions, it is clear that G is regular of degree $|X|$. We will also say that G has *valency* $|X|$. It is well known that G is connected if and only if $A = \langle X \rangle$. A *circulant* graph is a Cayley graph over a cyclic group $\mathbb{Z}/n\mathbb{Z}$. It shall be convenient to distinguish edges in a Cayley graph by their *edge type*, saying that an edge $e = \{a, a + x\}$ is an edge of *type* x. While an edge can simultaneously be a type x edge and a type $-x$ edge, we will be able to distinguish between types of *arcs*, an arc $(a, a + x)$ will be a type x arc, while $(a + x, a)$ will be a type $-x$ arc. For any Abelian Cayley graph $\mathbf{Cay}(A, S)$ and $x \in A$, the map $\phi_x : \mathbf{Cay}(A, S) \to \mathbf{Cay}(A, S)$ where $\phi_x(a) = a + x$ is a graph automorphism of $\mathbf{Cay}(A, S)$.

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A graph $\overline{G}$ is called a *covering* of G with *covering projection* $p : \overline{G} \to G$ if $p : V(\overline{G}) \to V(G)$ is a surjection such that $p|_{N(v')} : N(v') \to N(v)$ is a bijection for all $v \in V(G)$ and all $v' \in p^{-1}(v)$. This latter property is similar to the *local homeomorphism* property of coverings found in topology, but we will refer to it as the *local isomorphism property*. The set $p^{-1}(v)$ is called the *fiber* over v, and we will call p a *k-fold covering* if each fiber has size k. In this case, we will also say $\overline{G}$ is a *degree k* cover of G. The Cayley graph $\mathbf{Cay}(A, S)$ is $|S|$-regular, so $|S_1| = |S_2|$ is a necessary condition for $\mathbf{Cay}(A_1, S_1)$ to be a cover of $\mathbf{Cay}(A_2, S_2)$. Two coverings $p_1 : G_1 \to G$, $p_2 : G_2 \to G$ are said to be *isomorphic* if there is a graph isomorphism $\phi : G_1 \to G_2$ such that $p_1 = p_2 \circ \phi$. Such a ϕ is called a *covering isomorphism*.

Many times, graph covers are described and studied in terms of *voltage assignments*, as in [3,2]. The approach taken here will be different. Instead of looking at voltage assignments, we will concentrate on the structural properties of graphs dictated by the underlying group. The overall goal is to enumerate the isomorphism classes of connected Abelian covers of Abelian Cayley graphs. The results in this paper enumerate the isomorphism classes of connected circulant covers of connected circulant graphs of valency 3, which hereafter will be referred to as trivalent circulants.

2. Structural properties of Abelian covers

The main idea used in this paper to study Abelian covers of Abelian Cayley graphs is to look at what happens to the C_4 subgraphs. Fundamental to this study is the fact that if $p : \overline{G} \to G$ is a covering projection and H is a subgraph of $\overline{G}$ containing a C_4, then $p(H)$ contains a C_4 in G. Because this fact and its generalization are crucial in what follows, we provide proof of a more general assertion here.

Lemma 2.1. *Let $f : G_1 \to G_2$ be a covering projection. If $K_{n,m}$ is a complete bipartite subgraph of G_1, then $f(K_{n,m})$ must contain a $K_{n,m}$ in G_2. Further, if G_2 is triangle free, then $f(K_{n,m})$ must be an induced $K_{n,m}$.*

Proof. Let K be a complete bipartite subgraph of G_1, with bipartition $V = \{v_1, \dots, v_n\}$ and $W = \{w_1, \dots, w_m\}$. Now, for each v_i, we must have $N(v_i) \supseteq W$. Since f is a covering map, we must have $N(f(v_i)) \supseteq f(W)$. Similarly, for every w_j, we must have $N(f(w_j)) \supseteq f(V)$. Hence, the subgraph induced by $f(V)$ and $f(W)$ contains a complete bipartite subgraph, $K_{|f(V)|,|f(W)|}$. Since f is injective on V and W (as they are contained in the neighborhoods of w_1 and v_1, respectively), we have $|f(V)| = |V|$ and $|f(W)| = |W|$.If G_2 is triangle free, then none of the vertices in $f(V)$ can be adjacent, and none of the vertices in $f(W)$ can be adjacent. This completes the proof. □

Corollary 2.1. *If $f : G_1 \to G_2$ is a covering projection, then for each $R \cong C_4$ in G_1, $f(R)$ must contain a C_4 in G_2. If G_2 is triangle free, then $f(R)$ must in fact be a C_4.*

Proof. This follows immediately, since $C_4 \cong K_{2,2}$. □

Covering projections are a special type of graph homomorphism. There are a couple of facts about graph homomorphisms that we will need in the following.

Fact 2.1. *If P is a closed path in G_1, and $f : G_1 \to G_2$ is a graph homomorphism, then $f(P)$ is a closed path in G_2.*

Lemma 2.2. *If $f : G_1 \to G_2$ is a graph homomorphism, and G_2 is bipartite, then G_1 is bipartite.*

Proof. A graph being bipartite is equivalent to having a graph homomorphism to K_2. So, there is a graph homomorphism $\phi : G_2 \to K_2$. Then $\phi \circ f : G_1 \to K_2$ is a graph homomorphism, so G_1 is bipartite. □

We will also make use of the following.

Fact 2.2. *$G = \mathbf{Cay}(\mathbb{Z}/n\mathbb{Z}, \{\pm s_1, \dots, \pm s_t\})$ is bipartite $\iff$ $\gcd(2, s_i) = \gcd(2, n - s_i) = 1$ for all i.*

Rectangles in Abelian Cayley graphs: Abelian Cayley graphs have an abundance of C_4 subgraphs. This follows from the fact that in $G := \mathbf{Cay}(A, S)$, $S = \{\pm s_1, \dots, \pm s_n\}$, for $\alpha \in A$ and $i \neq j$, the subgraph induced by $\{\alpha, \alpha + s_i, \alpha + s_i +$

$s_j, \alpha+s_j\}$ contains a C_4. Essentially, this is saying that every commutator produces a collection of C_4's. For convenience, we will write $[a_0, a_1, a_2, a_3]$ to denote the directed cycle $a_0 \to a_1 \to a_2 \to a_3 \to a_0$. We will simultaneously refer to the C_4 defined by $[a_0, a_1, a_2, a_3]$ when talking about the subgraph $(\{a_0, a_1, a_2, a_3\}, \bigcup_{i=0}^{3}\{\{a_i, a_{i+1}\}_i\})$. This leads to the following definition.

Definition 2.1. Let $G := \mathbf{Cay}(A, S)$ be an Abelian Cayley graph. We will say that $R \cong C_4$ is an obvious rectangle if $R = [\alpha, \alpha + s, \alpha + s + s', \alpha + s']$ for some $\alpha \in A$, and some $s, s' \in S$with $s \neq \pm s'$.

Generally, there may be C_4 subgraphs in G which are not obvious rectangles. If G happens to not have any other C_4 subgraphs, then we will be able to describe the covers of G relatively easily. For this reason, we call G *rectangle rigid* if G has only obvious rectangles. One may observe that rectangle rigid graphs contain no $K_{3,2}$ subgraphs: in a rectangle rigid graph, any two incident edges which can be completed to a C_4 can only be completed to a C_4 in one way.

3. Covering circulants

We will begin by proving a couple of general facts about coverings which will be helpful for our classification of circulant coverings.

Lemma 3.1. *If $p : G_1 \to G_2$ is a covering projection, then for any $\alpha \in V(G_1)$, p maps the rectangles based at α injectively to rectangles based at $p(\alpha)$.*

Proof. By Lemma 2.1, the image of each rectangle must contain a rectangle. Suppose that R, R' are two rectangles based at $\alpha \in V(G_1)$ which map to the same rectangle based at $p(\alpha)$. Then $R \cap N[\alpha] = R' \cap N[\alpha]$, since p must map the edges incident with α in R and R' to the same edges incident with $p(\alpha)$. So, the vertex β at distance 2 from α in R must be different from the vertex β' at distance 2 from α in R'. But then $R \cup R'$ induces a $K_{2,3}$, so $p(R \cup R')$ must contain a $K_{2,3}$ by Lemma 2.1, contradicting the assumption that R and R' map to a single rectangle. See Fig. 1. This completes the proof. $\square$

Unless otherwise noted, we will use the following notation and will operate under the following hypotheses throughout this section: we will assume $G = \mathbf{Cay}(A, S)$, $G_1 = \mathbf{Cay}(A_1, S_1)$, $G_2 = \mathbf{Cay}(A_2, S_2)$, are connected circulants, $p : G_1 \to G_2$ is a covering projection, and $|S|, |S_1|, |S_2| \geqslant 3$. Since we are interested in determining covers up to isomorphism, we will use the following fact to assume $p(0_{A_1}) = 0_{A_2}$.

Fact 3.1. *If $p : \mathbf{Cay}(A_1, S_1) \to \mathbf{Cay}(A_2, S_2)$, is a covering projection with $p(x) = 0_{A_2}$ for some x, then p is isomorphic to $p \circ \phi_x$, and $p \circ \phi_x(0_{A_1}) = 0_{A_2}$.*

As the first step in classifying covers under these hypotheses, we will show that a rectangle rigid graph must be covered by a rectangle rigid graph. First, we will count how many obvious rectangles there are in circulant graphs.

Fact 3.2. *If $G = \mathbf{Cay}(A, S)$ is a circulant, then G has odd valency if and only if S contains an element of order 2. Since A is cyclic, A may have at most one order 2 element.*

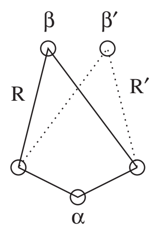

Fig. 1. Subgraph induced by R and R'.

Lemma 3.2. *If $G = \mathbf{Cay}(A, S)$ is circulant of even valency $\geqslant 4$, then G has* $\binom{\frac{|S|}{2}}{2} \cdot |A|$ *total obvious rectangles*, $4 \cdot \binom{\frac{|S|}{2}}{2}$ *obvious rectangles based at each vertex in G.*

Proof. An obvious rectangle from G where $S = \{\pm s_1, \ldots, \pm s_k\}$ consists of a choice of two indices from $\{1, \ldots, k\}$, a sign for each choice for each index and the choice of a base vertex. Each obvious rectangle is counted 4 times this way:

$$4 \cdot \binom{\frac{|S|}{2}}{2} \cdot \frac{|A|}{4} = \binom{\frac{|S|}{2}}{2} \cdot |A|. \qquad \square$$

A similar analysis can be done for circulants of odd order.

Lemma 3.3. *Let $G = \mathbf{Cay}(A, S)$ be a circulant of odd valency $\geqslant 3$:*

1. *If G has valency 3, then G has $|A|/2$ obvious rectangles.*
2. *If G has valency $\geqslant 5$, then G has* $\binom{\frac{|S|-1}{2}}{2} \cdot |A| + |A|(|S| - 1)/4$ *obvious rectangles.*

Proof. Let ω be the element of order 2 in S. For each pair of elements $\pm\alpha \in S$ with $\alpha \neq \omega$, there are $2|A|/4 = |A|/2$ obvious rectangles consisting of edges of type α and ω: choose a base vertex, and a sign. Each rectangle is counted 4 times this way.

1. If G has valency 3, then these are all of the obvious rectangles.
2. If G has valency $\geqslant 5$, then there are $(|S| - 1)/2$ pairs of elements $\pm\alpha \in S$ to use with ω for obvious rectangles, each type occurring $|A|/2$ times as above. By Lemma 3.2, we have $\binom{\frac{|S|-1}{2}}{2} \cdot |A|$ obvious rectangles not having any edges of type ω. Summing, we see that G has $\binom{\frac{|S|-1}{2}}{2} \cdot |A| + (|S| - 1)/2 \cdot |A|/2$ obvious rectangles. $\square$

Combining Lemmas 3.3 and 3.2 together, we see that the number of obvious rectangles in a covering graph is dictated by the degree of the cover along with the number of obvious rectangles in the base.

Lemma 3.4. *Let $p : G_1 \to G_2$ be a k-fold covering. If G_2 has N obvious rectangles, then G_1 has kN obvious rectangles. Also, the number of obvious rectangles based at a fixed vertex in G_1 is the same as the number of obvious rectangles based at a fixed vertex in G_2.*

Proof. Since the valency of G_1 and G_2 are the same, p being a k-fold covering implies that $|A_1| = k|A_2|$. The rest follows by inspecting the previous lemmas. $\square$

Corollary 3.1. *Let $p : G_1 \to G_2$ be a k-fold covering. If G_2 is rectangle rigid, then for any $\alpha \in A_1$, p maps the obvious rectangles based at α bijectively to the obvious rectangles based at $p(\alpha)$.*

Proof. By Lemma 3.1, the obvious rectangles are mapped injectively. By Lemma 3.4, this injection must be a bijection between the obvious rectangles. $\square$

We now have the machinery and definitions to prove the following important theorem.

Theorem 3.2. *Let $p : G_1 \to G_2$ be a k-fold covering. If G_2 is rectangle rigid, then G_1 is rectangle rigid.*

Proof. Suppose that G_1 is not rectangle rigid, so there is some non-obvious rectangle R in G_1 containing a vertex α. Now, the number of obvious rectangles based at α is the same as the number of obvious rectangles based at $p(\alpha)$.

Hence, there are more rectangles based at α than at $p(\alpha)$, which means p cannot map the rectangles based at α injectively to the rectangles based at $p(\alpha)$, contradicting Lemma 3.1. So, G_1 must also be rectangle rigid. $\square$

Note that the converse is not true. Let

$$G_1 = \mathbf{Cay}(\mathbb{Z}/30\mathbb{Z}, \{\pm 1, \pm 7\}), \quad G_2 = \mathbf{Cay}(\mathbb{Z}/5\mathbb{Z}, \{\pm 1, \pm 2\}) \cong K_5.$$

Then the normal group quotient from $\mathbb{Z}/30\mathbb{Z} \to \mathbb{Z}/5\mathbb{Z}$ induces a covering map, but G_1 is rectangle rigid and triangle free, while G_2 is not. This theorem simplifies matters when looking for covers of a rectangle rigid circulant: we need only to look at other rectangle rigid circulants. With this in hand, we will show coverings between rectangle rigid graphs induce a map between the edge types. In fact, we will show something slightly stronger. Recall, (u, v) denotes the arc from u to v, an arc of type $v - u$. We will write $\mathbf{type}(u, v)$ to denote the edge type of (u, v).

Theorem 3.3. *Suppose that $G_1 = \mathbf{Cay}(A_1, S_1)$ and $G_2 = \mathbf{Cay}(A_2, S_2)$ are rectangle rigid graphs, and $f : G_1 \to G_2$ a k-fold covering. Then there exists a map $\overline{f} : S_1 \to S_2$ such that each arc of type $\alpha \in S_1$ is mapped to an arc of type $\overline{f}(\alpha)$.*

Proof. By applying a suitable rotation, we may assume that $f(0)=0$. So, we will prove the following claim by induction on d, the distance of the vertex v from 0: for all $w \in N(v)$, $\mathbf{type}(f((v, w))) = \mathbf{type}(f((0, w - v)))$. Note for $s \in S$, $\mathbf{type}(f((0, s))) = -\mathbf{type}(f((0, -s)))$: if the path $(-s, 0, s)$ is mapped to a path consisting of two different types of arcs, say $(\alpha, 0, \beta)$, then the obvious rectangle in G_2 consisting of $(\alpha, 0, \beta)$ in G_2 will not be mapped to by any obvious rectangle based at 0 in G_1, contradicting Lemma 3.1. So, $\mathbf{type}(f((0, s))) = -\mathbf{type}(f((0, -s)))$ for all $s \in S$. For $d=0$, the claim is clear. So, assume that $d > 0$, and let v be a vertex distance d from 0. There is some v' such that $v' \in N(v)$ and v' is distance $d - 1$ from 0. By induction, the type of $f((v', v))$ is the same as the type of $f((0, v - v'))$. For every vertex $w \in N(v')$ with $\mathbf{type}(v', w) \neq \pm\mathbf{type}(v', v)$, there is a unique rectangle containing the vertices w, v', v, the obvious rectangle $R := [w, v', v, v + w - v']$. Since this R is obvious, we have $\mathbf{type}(v, v + w - v') = -\mathbf{type}(w, v') = \mathbf{type}(v', w)$. By Theorem 3.2, $f(R)$ must be an obvious rectangle, so $\mathbf{type}(f((v, v+w-v'))) = -\mathbf{type}(f((w, v'))) = \mathbf{type}(f((v', w)))$, which by induction is $\mathbf{type}(f((0, w - v'))) = \mathbf{type}(f((0, (v + w - v') - v)))$. So, if (v, z) is an arc out of v of type not equal to $\pm\mathbf{type}(v, v')$, we have $\mathbf{type}(f((v, z))) = \mathbf{type}(f((0, z - v)))$. Also, $\mathbf{type}(f((v, v'))) = -\mathbf{type}(f((v', v))) = -\mathbf{type}(f((0, v - v'))) = \mathbf{type}(f((0, v' - v)))$. So, there is only one other arc type out of v, the arc $(v, v + (v - v'))$. This must be mapped to the remaining edge type out of $f(v)$, which is $\mathbf{type}(f((0, v - v')))$. See Fig. 2. Hence, for all $w \in N(v)$, we have $\mathbf{type}(f((v, w))) = \mathbf{type}(f((0, w - v)))$ which completes the proof of the theorem. $\square$

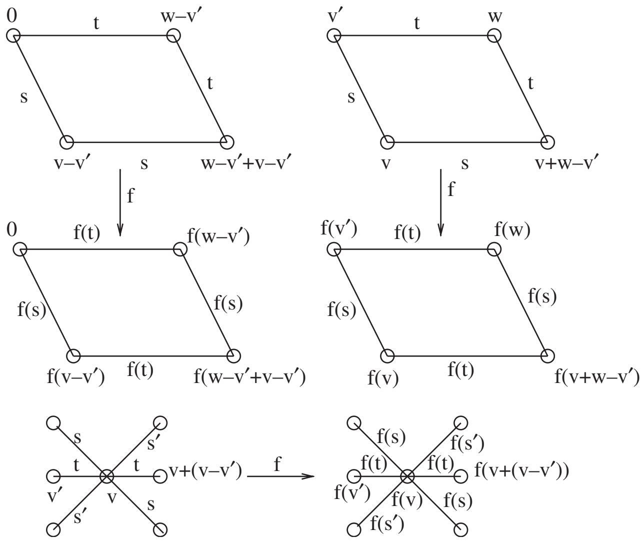

Fig. 2. Neighborhood of v inductive step.

We have constantly made the assumption that 0 is mapped to 0 in our maps. Let f be a covering projection. We will call f *normalized* if $f(0)=0$. Further, we will call g a *normalization of* f if $g=f\circ\phi_x$ for some ϕ_x, and g is normalized. We will use some group theory to help us show that all of the covering maps which preserve the edge type have normalizations which are group homomorphisms. The key is the following. See [4].

Theorem 3.4 (*Van Dyck*). *Let X be a set, $\mathscr{C}$ a (possibly infinite) set of reduced words on X, and G the group with presentation $\langle X|\mathscr{C}\rangle$. Let H be any group, and $f: X\to H$ an injective set map. If $W=w_1\dots w_n$ a word over X, then by $f(W)$ we mean the word $f(w_1)\dots f(w_n)$ over $f(X)$. If $H=\langle f(X)\rangle$ and H satisfies all of the relations $\mathscr{C}'=\{f(C)|C\in\mathscr{C}\}$, then f extends uniquely to a surjective group homomorphism $\overline{f}: G\to H$.*

The idea is to use a correspondence between the cycles in the graph and the relations for $\mathscr{C}$. Since covering projections must map cycles to cycles, then the target group will satisfy the relations in $\mathscr{C}'$. This gives us the following.

Theorem 3.5. *Let $G_1=\mathbf{Cay}(A_1,S_1)$ and $G_2=\mathbf{Cay}(A_2,S_2)$. Suppose $p: G_1\to G_2$ is a k-fold covering such that there is a map $\overline{p}: S_1\to S_2$ such that for each $s\in S_1$, every arc of type s is mapped to an arc of type $\overline{p}(s)$. If p^* is a normalization of p, then p^* is a group homomorphism from A_1 to A_2.*

Proof. Suppose $S_1=\{\pm s_1,\dots,\pm s_m\}$. Take $S_1^+=\{s_1,\dots,s_m\}$. By assumption, every arc of type s_i is mapped to an arc of type $\overline{p}(s_i)$, thus every arc of type $-s_i$ is mapped to an arc of type $-\overline{p}(s_i)$. If p^* is a normalization of p, then p^* is injective on S_1^+, as $S_1^+\subset N(0)$. Let $\mathscr{C}$ be the set of reduced words over S_1 which form closed paths in G_1. Since S_1 generates A_1 as a group, and every relation in A_1 corresponds to some closed path in G_1, we see that $\langle S_1^+,\mathscr{C}\rangle$ is a presentation for A_1. Any normalization of p, say p^*, gives a well-defined injective map from words over S_1 to words over S_2. Note $\overline{p}(s_i)=p^*(s_i)$. Now, since p^* is a covering map, p^* must take closed paths to closed paths, which in turn must be relations in the group A_2. So, A_2 satisfies the relations $\mathscr{C}'=\{\overline{p}(C)|C\in\mathscr{C}\}$. So, $\overline{p}|_{S_1^+}=p^*|_{S_1^+}$ satisfies the hypotheses for the Van Dyck theorem, which implies that $p^*|_{S_1^+}$ must extend to a group homomorphism f. Since f and p^* agree on words over S_1, we must have p^* is a group homomorphism. □

Corollary 3.6. *Let $G_1=\mathbf{Cay}(A_1,S_1)$, $G_2=\mathbf{Cay}(A_2,S_2)$. If $p: G_1\to G_2$ is a covering map with G_1 and G_2 rectangle rigid, then p is isomorphic to a group homomorphism.*

Proof. By Theorem 3.3, there is an $\overline{p}: S_1\to S_2$ so that for each $\alpha\in S_1$, every arc of type α is mapped to an arc of type $\overline{p}(\alpha)$. Further, by Theorem 3.5, any normalization of p is a group homomorphism. Since p is isomorphic to all of its normalizations, we are done. □

4. Applications

Now that we have some machinery in place, we will be able to classify all covers for some low valency circulants. We spend the remainder of the paper classifying connected circulant covers of connected trivalent circulants. Note that all connected trivalent circulants are of the form $G=\mathbf{Cay}(\mathbb{Z}/2n\mathbb{Z},\{\pm\alpha,n\})$ for some α with $\gcd(\alpha,n)=1$. So, $\gcd(\alpha,2n)\in\{1,2\}$. The following fact can be observed easily.

Fact 4.1. *If $G=\mathbf{Cay}(\mathbb{Z}/2n\mathbb{Z},S)$ is a trivalent circulant of order $2n\geqslant 8$, then G is rectangle rigid.*

Since the trivalent circulants of order $\geqslant 8$ are rectangle rigid, they can only have covers which can be normalized to group homomorphisms. So, we need to only inspect which possible group homomorphisms can be used as covers, and which are the same up to isomorphism. Separately, we will have to classify the covers of $\mathbf{Cay}(\mathbb{Z}/6\mathbb{Z},\{\pm1,3\})\cong K_{3,3}$, and the covers of $\mathbf{Cay}(\mathbb{Z}/4\mathbb{Z},\{\pm1,2\})\cong K_4$, both of which are not rectangle rigid circulants.

Circulants of order $\geqslant 8$: All of the covering projections over circulants of order $\geqslant 8$ must be isomorphic to group homomorphisms. We include the analysis for completeness. Since all covering projections are isomorphic to group homomorphisms, we will operate under the assumption each covering projection is a group homomorphism.

If $G=\mathbf{Cay}(\mathbb{Z}/2n\mathbb{Z},\{\pm\alpha,n\})$ is a circulant with $n\geqslant 4$, then G is rectangle rigid. Hence, every normalized k-fold covering $p: G':=\mathbf{Cay}(\mathbb{Z}/2kn\mathbb{Z},\{\pm\beta,kn\})\to G$ is a group homomorphism. If k is even, then kn

must be in the kernel of p, which contradicts p being a covering projection. So, k must be odd. Recall, gcd $(\beta, 2n) \in \{1, 2\}$.

1. If $n = 2m$ is even, then $\gcd(\alpha, 2m) = 1 \Rightarrow \gcd(\alpha, 2n) = 1$. Similarly, $\gcd(\beta, 2km) = 1 \Rightarrow \gcd(\beta, 2) = 1$. By counting, we know that the edges of type β in G' must be mapped to edges of type α in G. Now, either $p(t\beta) = t\alpha$ or $p(t\beta) = -t\alpha$ for all $t \in \mathbb{Z}$. Look at $p_1 : G' \to G$, where $p_1(t\beta) = t\alpha$ and $p_2 : G' \to G$, where $p_2(t\beta) = -t\alpha$. Note that $\tau(t\beta) = -t\beta$ is a graph automorphism of G'. Then $p_1 \circ \tau(t\alpha) = p_1(-t\alpha) = -t\beta = p_2(t\alpha)$. Hence, p_1 and p_2 are isomorphic. If $G_0 = \mathbf{Cay}(\mathbb{Z}/2kn\mathbb{Z}, \{\pm\beta', kn\})$ is any other graph with $p_0 : G_0 \to G$ a k-fold covering, since $\gcd(\beta', 2km) = \gcd(\beta, 2km) = 1$, there is a graph isomorphism $\phi : G_0 \to G'$ such that all edges of type β' are mapped to edges of type β, hence $p_0 \in \{p_1 \circ \phi, p_2 \circ \phi\}$. So, for each k odd, there is exactly one isomorphism class for connected circulant covers over graphs of this type.
2. If n is odd, then either $G \cong \mathbf{Cay}(\mathbb{Z}/2n\mathbb{Z}, \{\pm 1, n\})$, or $G \cong \mathbf{Cay}(\mathbb{Z}/2n\mathbb{Z}, \{\pm 2, n\})$. If $G \cong \mathbf{Cay}(\mathbb{Z}/2n\mathbb{Z}, \{\pm 1, n\})$, then G is bipartite, so G' must be bipartite. Hence, $\gcd(\beta, 2) = 1$. As before, for each k odd, we see there is one isomorphism class over graphs G of this type. If $G \cong \mathbf{Cay}(\mathbb{Z}/2n\mathbb{Z}, \{\pm 2, n\})$, then by counting, we know that all edges of type kn must be mapped to edges of type n, so the restriction of p to $\mathbf{Cay}(\mathbb{Z}/2kn\mathbb{Z}, \{\pm\beta\})$ gives a covering of $\mathbf{Cay}(\mathbb{Z}/2n\mathbb{Z}, \{\pm\alpha\}) \cong C_n \sqcup C_n$, the disjoint union of two n-cycles. So $\mathbf{Cay}(\mathbb{Z}/2kn\mathbb{Z}, \{\pm\beta\})$ must contain two kn cycles, hence $\gcd(\beta, 2n) = 2$. Again, following the analysis as before, for each k odd, we see there is one isomorphism class over graphs G of this type.

This proves the following.

Theorem 4.1. *If G is a trivalent circulant of order $\geqslant 8$, then there are no circulant k-fold coverings of G for even k, and for every odd k, there is a unique connected circulant k-fold covering of G up to isomorphism.*

For the later sections, it will be useful to know the automorphism groups of these trivalent circulants, so we will state that here. Since the case we need is a corollary of our results, we give its simple proof here.

Theorem 4.2. *For $n \geqslant 4$, if $G = \mathbf{Cay}(\mathbb{Z}/2n\mathbb{Z}, \{\pm\alpha, n\})$, with $\gcd(\alpha, 2n) = 1$, then $\mathbf{Aut}(G) \cong D_{2n}$, the automorphism group of the regular $2n$-gon.*

Proof. Note that 1-coverings are graph isomorphisms. Let g be an automorphism of G. By counting, since G is rectangle rigid, type n edges must be mapped to type n edges. So, g restricted to edges of type α gives an automorphism of C_{2n}. So, $\mathbf{Aut}(G) \subseteq D_{2n}$. One can check that the maps $\phi_1(x) = x + 1$ and $\tau(x) = -x$ are automorphisms of G which generate a subgroup isomorphic to D_{2n}, so $\mathbf{Aut}(G) \cong D_{2n}$. $\square$

Covers of $K_{3,3}$: Because of the rectangle rigidity, most of the circulant covers of trivalent circulants can be described relatively easily. The only remaining cases which we need to examine are circulant covers of $\mathbf{Cay}(\mathbb{Z}/6\mathbb{Z}, \{\pm 1, 3\}) \cong K_{3,3}$ and $\mathbf{Cay}(\mathbb{Z}/4\mathbb{Z}, \{\pm 1, 2\}) \cong K_4$. We will start with $K_{3,3}$. For the remainder of this section, $G = \mathbf{Cay}(\mathbb{Z}/2n\mathbb{Z}, \{\pm\alpha, n\})$. If G is a cover of $K_{3,3}$, we can observe the following. Since $K_{3,3}$ is bipartite, we must have G bipartite, so n must be odd, and α must also be odd. Since $\gcd(\alpha, n, 2n) = 1 = \gcd(\alpha, n)$ and α is odd, $\gcd(\alpha, 2n) = 1$. So, we have $G \cong \mathbf{Cay}(\mathbb{Z}/2n\mathbb{Z}, \{\pm 1, n\})$. If G is a k-covering of $K_{3,3}$, then $6k = 2n \Rightarrow 3k = n$. Since n is odd, k cannot be even. Hence, G cannot be a k-covering for any even k.

Lemma 4.1. *If $f : G \to K_{3,3}$ is a k-covering, then f must take the edges of type n to a perfect matching in $K_{3,3}$.*

Proof. Let $K_{3,3}$ have bipartition $A = \{a_0, a_1, a_2\}$, $B = \{b_0, b_1, b_2\}$. As described above, $\gcd(\alpha, 2n) = 1$. Then without loss of generality, we may assume that $\alpha = 1$ (by applying a suitable isomorphism). Consider the subgraph induced by $\{-1, 0, 1, n-1, n, n+1\}$ in G. Since $[-1, n-1, n, 0]$ forms a rectangle in G, it must be mapped to a rectangle in $K_{3,3}$, hence the two edges of type n in this rectangle must be mapped to independent edges. By applying a suitable automorphism of $K_{3,3}$, we will assume $\{-1, n-1\}$ maps to $\{a_1, b_1\}$ and $\{0, n\}$ maps to $\{a_2, b_2\}$ $(f(-1) = a_1, f(0) = b_2, f(n-1) = b_1, f(n) = a_2)$.Now, since the neighborhood of 0 must map bijectively to the neighborhood of b_2, we must have $f(1) = a_3$. Similarly, the neighborhood of n must map bijectively to the neighborhood of a_2, so we must

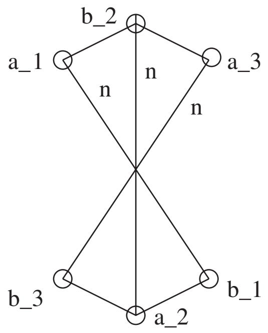

Fig. 3. Circulant covering of $K_{3,3}$.

have $f(n+1)=b_3$. See Fig. 3. So, we now have the following statement:

$$\left\{\begin{array}{c} f(-1)=a_0 \\ f(n-1)=b_0 \\ f(0)=b_1 \\ f(n)=a_1 \end{array}\right\} \Rightarrow \left\{\begin{array}{c} f(1)=a_2 \\ f(n+1)=b_2 \end{array}\right\}.$$

Applying the same argument repeatedly, we will have

$$\left\{\begin{array}{c} f(k-1)=a_{j-1} \\ f(n+k-1)=b_{j-1} \\ f(k)=b_j \\ f(n+k)=a_j \end{array}\right\} \Rightarrow \left\{\begin{array}{c} f(k+1)=a_{j+1} \\ f(n+k+1)=b_{j+1} \end{array}\right\},$$

where all subscripts are taken modulo 3. Hence, every edge of type n will go to an edge of the form $\{a_i, b_i\}$, which form a matching in $K_{3,3}$. $\square$

Now given any matching M in $K_{3,3}$, we will show there is a covering of G' over $K_{3,3}$ which takes the edges of type n to the matching M. Note that we need only prove that there is a covering of G' over $K_{3,3}$ which takes the edges of type n to a specific perfect matching M'. Since the automorphism group of $K_{3,3}$ acts transitively on perfect matchings and coverings composed with automorphisms are still coverings, if we can prove the result for some specific perfect matching, then we will be done.

Theorem 4.3. *Suppose that M is a matching in $K_{3,3}$. Then for all $k \geqslant 1$ and $1 \leqslant \alpha < 3k$ with* $\gcd(\alpha, 6k)=1$, *there is some covering $f: G \to K_{3,3}$ such that the set of type $3k$ edges are mapped to M. Further, every covering mapping type $3k$ edges to M maps type α edges to type 1 edges.*

Proof. As described above, we need only prove that there is some covering that takes the set of type $3k$ edges to a specific matching, so choose the matching consisting of type 3 edges while viewing $K_{3,3}$ as the Cayley graph $\mathbf{Cay}(\mathbb{Z}/6\mathbb{Z}, \{\pm 1, 3\})$. If $k \geqslant 1$ odd, then it is enough to observe that the map $p: \langle\alpha\rangle \to \mathbb{Z}/6\mathbb{Z}$, given by $p(t\alpha)=t \pmod 6$ maps type α edges to type 1 edges, and type $3k$ edges to type $3k \equiv 3 \pmod 6$ edges (as k odd). So we do in fact have a covering map. The latter claim follows, since every covering which maps all the $3k$ edges to M must take the type α edges to the other edges, which are type 1. $\square$

Corollary 4.4. *Let $G' = \mathbf{Cay}(\mathbb{Z}/6k\mathbb{Z}, \{\pm\alpha', 3k\})$, $G = \mathbf{Cay}(\mathbb{Z}/6k\mathbb{Z}, \{\pm\alpha, 3k\})$. If $f': G' \to K_{3,3}$ if a k-covering and $f: G \to K_{3,3}$ is another k-covering such that f and f' both map type $3k$ edges to M, then f and f' are isomorphic.*

Proof. Since α and α' have $\gcd(\alpha, 6k)=\gcd(\alpha', 6k)=1$, there is a graph isomorphism from $\phi: G \to G'$ taking all edges of arcs of type α to arcs of type α'. Specifically, $\pi_\beta: \mathbf{Cay}(\mathbb{Z}/6k\mathbb{Z}, \{\pm 1, 3k\} \Rightarrow \mathbf{Cay}(\mathbb{Z}/6k\mathbb{Z}, \{\pm\beta, 3k\})$ given by $\pi_\beta(t)=\beta t$ is a graph isomorphism, so we can take $\phi = \pi_{\alpha'} \circ (\pi_\alpha)^{-1}$. Now, the subgraph of G consisting of edges of type α must be a covering of C, a 6-cycle $K_{3,3}$. Label the vertices of C $\{0', 1', \ldots, 5'\}$such that $f(t\alpha) = t'$, where t is reduced modulo 6. Similarly, the subgraph of G' consisting of edges of type α' must be a covering of C. Then, $f' \circ \phi_s(0) = 0'$ for some $s \in \mathbb{Z}/6k\mathbb{Z}$. Hence, either $f' \circ \phi_s(t\alpha') = t'$ or $f' \circ \phi_s(t\alpha') = -t'$. Since $\tau(x) = -x$ is an automorphism of G', f' is isomorphic to $g : \mathbf{Cay}(\mathbb{Z}/6k\mathbb{Z}, \{\pm\alpha', 3k\}) \to K_{3,3}$, where $g(t\alpha') = t'$. Then $f = g \circ \phi$, so, f and f' are isomorphic. $\square$

Next, we need to observe that different matchings in the base graph give rise to different covers.

Lemma 4.2. *Suppose that M_1 and M_2 are two different perfect matchings in $K_{3,3}$. Suppose we have two k-coverings $f_1, f_2 : \mathbf{Cay}(\mathbb{Z}/6k\mathbb{Z}, \{\pm 1, 3k\}) \to K_{3,3}$ such that f_i takes the set of type $3k$ edges to M_i. Then f_1 and f_2 cannot be isomorphic for $k > 1$.*

Proof. By taking a normalization of f_1 and f_2, we may assume that $f_i(0) = 0$ (viewing $K_{3,3}$ as $\mathbf{Cay}(\mathbb{Z}/6\mathbb{Z}, \{\pm 1, 3\})$). Next, note that the edges $\{0, n\}, \{2n-1, n-1\}, \{1, n+1\}$ must be mapped bijectively to M_1 and M_2 by f_1 and f_2, respectively. If $k > 1$, then $\mathbf{Aut}(\mathbf{Cay}(\mathbb{Z}/6k\mathbb{Z}, \{\pm 1, 3k\})) \cong D_{6k}$. These automorphisms preserve edge type, so there is no automorphism ϕ such that $f_1 \circ \phi = f_2$ and vice versa, so f_1 and f_2 must belong to different isomorphism classes. $\square$

Corollary 4.5. *For each odd $k > 1$, there are six isomorphism classes of connected circulant k-coverings of $K_{3,3}$, one for each perfect matching in $K_{3,3}$. For each even k, there are no circulant k-coverings of $K_{3,3}$.*

So, we have produced a classification of all circulant coverings of $K_{3,3}$.

Covers of K_4: The situation for circulant coverings of K_4 is a little more delicate. First, K_4 is not rectangle rigid, and second, K_4 is not bipartite. This does allow the structure of the covering graph to be a little more varied than in the rectangle rigid case, or in the case of covers of $K_{3,3}$. Fortunately, the assumption that the covering graph is circulant does impose quite a bit of restriction on what the covering maps may look like. We will use K_4 and $\mathbf{Cay}(\mathbb{Z}/4\mathbb{Z}, \{\pm 1, 2\})$ interchangeably in what follows. The advantage to using the Cayley representation is that while K_4 has no distinguishing features between edges, the Cayley graphs has two types of edges, namely type 1 and type 2 edges. This will be important in the following. Again, any circulant covering of K_4 must be of the form $\mathbf{Cay}(\mathbb{Z}/2n\mathbb{Z}, \{\pm\alpha, n\})$.

Lemma 4.3. *If $G = \mathbf{Cay}(\mathbb{Z}/4k\mathbb{Z}, \{\pm\alpha, 2k\})$ is connected, then $G \cong \mathbf{Cay}(\mathbb{Z}/4k\mathbb{Z}, \{\pm 1, 2k\})$.*

Proof. For $\mathbf{Cay}(\mathbb{Z}/4k\mathbb{Z}, \{\pm\alpha, 2k\})$ to be connected, we must have $\gcd(\alpha, 2k, 4k) = 1$. Hence, $\gcd(\alpha, 2k) = \gcd(\alpha, 4k) = 1$. $\square$

Lemma 4.4. *Suppose that $f : \mathbf{Cay}(\mathbb{Z}/2n\mathbb{Z}, \{\pm\alpha, n\}) \to K_4$ is a k-covering. Either all type n edges are mapped to type 1 edges or all type n edges are mapped to type 2 edges.*

Proof. We will show that either all type n edges are mapped to type 2 edges, or none of them are. Suppose that the arc $(\beta, \beta+n)$ is mapped to the arc $(0, 2)$. Then, the rectangle $[\beta, \beta+n, \beta+n+\alpha, \beta+\alpha]$ must be mapped to an obvious rectangle, since type 2 edges of $\mathbf{Cay}(\mathbb{Z}/4\mathbb{Z}, \{\pm 1, 2\})$ only belong to obvious rectangles. Hence, the edge $\{\beta+\alpha, \beta+\alpha+n\}$ must also be mapped to an edge of type 2. By induction, $\{\beta+k\alpha, \beta+k\alpha+n\}$ must be mapped to a type 2 edge for all $k \geqslant 0$. Since this accounts for all edges of type n, we can conclude if a single edge of type n is mapped to a type 2 edge, then all edges of type n are mapped to type 2 edges. $\square$

This leaves us to distinguish between the two possibilities for edges of type n.

Theorem 4.6. *Suppose that $f : \mathbf{Cay}(\mathbb{Z}/4k\mathbb{Z}, \{\pm\alpha, 2k\}) \to K_4$ is a k-covering which takes all type $2k$ edges to type 2 edges. Then f is isomorphic to a group homomorphism from $\mathbb{Z}/4k\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$.*

Proof. By the local isomorphism property, we see no type α edges are mapped to type 2 edges. This means that the restriction of f to $\mathbf{Cay}(\mathbb{Z}/4k\mathbb{Z}, \{\pm\alpha\})$ is a k-covering of C_4. So, all arcs of type α must be mapped to arcs of type $f(\alpha)$. Then by Theorem 3.5, f must be isomorphic to a group homomorphism. $\square$

Corollary 4.7. *If k is even, then there are no circulant k-coverings of K_4 taking type $2k$ edges to type 2 edges.*

Proof. If p is a covering of this type, then p is isomorphic to a group homomorphism. If k is even, then $4|2k$, hence $2k$ is in the kernel of any homomorphism from $\mathbb{Z}/4k\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$. So, type $2k$ edges would collapse to loops, which does not give a valid covering of K_4. $\square$

The other option is that all of type n edges are mapped to type 1 edges. Let $G = \mathbf{Cay}(\mathbb{Z}/4k\mathbb{Z}, \{\pm\alpha, 2k\})$, and let $f : G \to K_4$ be a k-covering. For each vertex $v \in V(G)$, exactly one of the type α edges incident with v must map to an edge of type 2 in K_4 by the local isomorphism property. Hence, the edges in G mapping to edges of type 2 must form a perfect matching in the subgraph consisting of type α edges. This observation will help classify the rest of the circulant covers of K_4.

Theorem 4.8. *Suppose that $k \geqslant 2$. If k is odd, then there are only two isomorphism classes of k-coverings in which the type $2k$ edges are mapped to type 1 edges. There is a map f_+ where $f_+(\{0, 2k\}) = \{0, 1\}$ which is the representative for one isomorphism class, and a map f_- where $f_-(\{0, 2k\}) = \{0, 3\}$ which is the representative for the other isomorphism class. If k is even, then there are no k-coverings with type $2k$ edges mapped to type 1 edges.*

Proof. As before, the set of edges in $\mathbf{Cay}(\mathbb{Z}/4k\mathbb{Z}, \{\pm 1\})$ which map to edges of type 2 form a perfect matching. By applying a suitable rotation, we may assume the perfect matching consists of the edges $M := \{\{2i, 2i + 1\}| i = 0, \ldots, 2k - 1\}$. Now, the rectangles in $\mathbf{Cay}(\mathbb{Z}/4k\mathbb{Z}, \{\pm 1, 2k\})$ are of the form $[i, i + 1, i + 1 + 2k, i + 2k]$. Note that the edges of type 1 in a rectangle are either both in M or both not in M. The rest of the proof will consist of two steps. First, if f, g are two k-coverings in which type $2k$ edges are mapped to type 1 edges, M is mapped to type 2 edges and the arc $(0, 2k)$ is mapped to the arc $(0, 1)$ by both f and g, then $f = g$. Second, we will show there is a such a map f.

Suppose that f, g are as described. We will prove the following claim by induction on β: $f|_{N[\beta]} = g|_{N[\beta]}$ and $f|_{N[\beta+2k]} = g|_{N[\beta+2k]}$ for $0 \leqslant \beta < 2k$. By assumption, at $\beta = 0$, we have $f(\beta) = g(\beta) = 0$, $f(\beta + 2k) = g(\beta + 2k) = 1$. Now, by assumption, both f and g map edges from M to edges of type 2, so $f(\beta + 1) = g(\beta + 1) = 2$. By the local isomorphism property, we must have $f(\beta - 1) = g(\beta - 1) = 3$. Since the rectangle $[0, 1, 1 + 2k, 2k]$ must be mapped to a rectangle, we must have $f(\beta + 2k + 1) = g(\beta + 2k + 1) = 3$. Similarly, the neighborhood bijection property at $\beta + 2k$ tells us that $f(\beta + 2k - 1) = g(\beta + 2k - 1) = 2$. So, the claim holds at $\beta = 0$.

Suppose now that $\beta + 1 > 0$. Now, $N[\beta + 1] - (N[\beta] \cup N[\beta + 2k]) = \{\beta + 2\}$. By induction, f and g agree on $N[\beta] \cup N[\beta + 2k]$, so by the neighborhood bijection property, they must map $\beta + 2$ to the same place, so f and g agree on $N[\beta + 1]$. Similarly, $N[\beta + 2k + 1] - (N[\beta + 2k] \cup N[\beta]) = \{\beta + 2k + 2\}$, so f and g agree on $N[\beta + 2k + 1]$. Hence, $f = g$.

Note that using this argument, one can show $f((0, 2k)) = (0, 1)$, $f((1, 2k + 1)) = (2, 3)$, and generally:

$$f((\beta, \beta + 2k)) = \begin{cases} (0, 1) : \beta \equiv 0 \pmod 4, \\ (2, 3) : \beta \equiv 1 \pmod 4, \\ (1, 0) : \beta \equiv 2 \pmod 4, \\ (3, 2) : \beta \equiv 3 \pmod 4. \end{cases}$$

If k is even, then $2k \equiv 0 \pmod 4$, so the arc $(0, 2k)$ maps to the arc $(0, 1)$ but the arc $(2k, 0)$ also maps to the arc $(0, 1)$, a contradiction. So, there is no k-covering with these properties for k even. A similar argument shows there is no k-covering mapping M to type 2 edges and $(0, 2k)$ to $(0, 3)$ for k even.

Now, we need to show that such a map f exists for k odd. We will define $f_+(t)$ as follows:

$$f_+(t) = \begin{cases} 0 : t \equiv 0 \pmod 4, \\ 2 : t \equiv 1 \pmod 4, \\ 1 : t \equiv 2 \pmod 4, \\ 3 : t \equiv 3 \pmod 4. \end{cases}$$

See Fig. 4. Let $\beta \in \mathbb{Z}/4k\mathbb{Z}$. If k is odd, then $2k \equiv 2 \pmod 4$. Hence, the closed neighborhood of β must include one element in each residue class modulo 4, which shows the local isomorphism property holds. Hence, f_+ is a k-covering

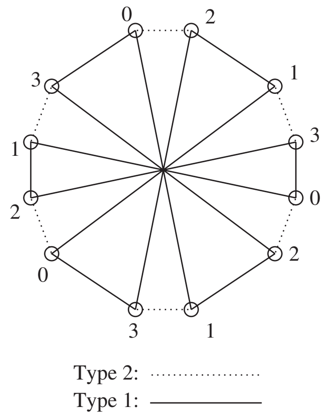

Fig. 4. Type n to type 1 covering of K_4.

of K_4. A similar argument shows that there is a unique f_- up to isomorphism with

$$f_-(k) = \begin{cases} 0 : k \equiv 0 \pmod 4, \\ 2 : k \equiv 1 \pmod 4, \\ 3 : k \equiv 2 \pmod 4, \\ 1 : k \equiv 3 \pmod 4. \end{cases}$$

Note f_+ and f_- cannot be isomorphic: $\mathbf{Aut}(\mathbf{Cay}(\mathbb{Z}/4k\mathbb{Z}, \{\pm 1, 2k\})) \cong D_{4k}$, so there is no automorphism which will map the type $2k$ edges originally mapped to $\{0, 1\}$ and $\{2, 3\}$ under f_+ to the type 1 edges mapped to $\{0, 1\}$ and $\{2, 3\}$ under f_-. This completes the proof. □

5. Conclusion

By using the C_4 structure within circulant graphs, we are able to describe the isomorphism classes of circulant covers over trivalent circulants in most cases. Notably, there are no even fold circulant covers of any trivalent circulant. Only a couple of sporadic examples required separate attention. One should be able to use a similar approach to determine the isomorphism classes of circulant covers over valency 4 circulants. For higher valency, more general techniques would be of interest.

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